1. (a) Express $\dfrac{3}{(2x-1)(x+1)}$ in partial fractions.

(3)

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, $V\text{m}^3$, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \qquad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of $3\,\text{m}^3$ of oxygen had been produced,

(b) solve the differential equation to show that

$$V = \frac{3(2t-1)}{(t+1)}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes,

(ii) the **limit** giving your answer in m^3

(2)

a) $\dfrac{3}{(2x-1)(x+1)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{x+1}$ 1

$3 \equiv A(x+1) + B(2x-1)$

$x = -1$: $3 = -3B$ $\therefore$ $B = -1$ 1

$x = \frac{1}{2}$: $3 = \frac{3}{2}A$ $\therefore$ $A = 2$

$\therefore \dfrac{3}{(2x-1)(x+1)} \equiv \dfrac{2}{2x-1} - \dfrac{1}{x+1}$ 1

b) $\frac{dV}{dt} = \frac{3V}{(2t-1)(t+1)}$, $V \geq 0$, $t \geq k$

$$\int \frac{1}{V}\, dV = \int \frac{3}{(2t-1)(t+1)}\, dt \quad (1)$$

$$= \int \left\{ \frac{2}{2t-1} - \frac{1}{t+1} \right\} dt \quad (1)$$

$$\ln|V| = \ln|2t-1| - \ln|t+1| + c \quad (1)$$

$$\ln|V| = \ln\left|\frac{2t-1}{t+1}\right| + c$$

when $t = 2$ and $V = 3$,

$$\ln 3 = \ln\left(\frac{3}{3}\right) + c$$

$$\ln 3 = \ln 1 + c$$

$$\therefore c = \ln 3 \quad (1)$$

$$\therefore \ln|V| = \ln\left|\frac{2t-1}{t+1}\right| + \ln 3$$

$$\ln|V| = \ln\left|\frac{3(2t-1)}{t+1}\right|$$

$$\therefore V = \frac{3(2t-1)}{t+1} \quad (1)$$

(c) (i) $V = 0$: $\frac{3(2t-1)}{t+1} = 0$

$2t - 1 = 0 \quad \therefore t = \frac{1}{2}$

$\therefore$ 30 minutes (1)

(ii) $V = \frac{6t-3}{t+1}$

$V = \frac{6 - \frac{3}{t}}{1 + \frac{1}{t}}$

when $t \to \infty$, $V = \frac{6-0}{1+0} = 6$

$\therefore$ limit is $6\,m^3$ (1)

2.

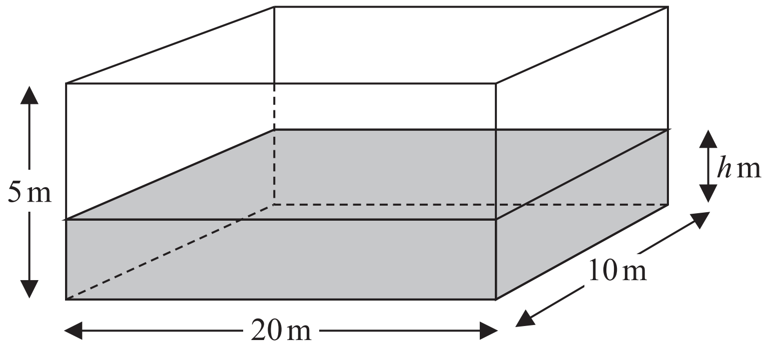

Figure 1

A tank in the shape of a cuboid is being filled with water.

The base of the tank measures 20 m by 10 m and the height of the tank is 5 m, as shown in Figure 1.

At time t minutes after water started flowing into the tank the height of the water was h m and the volume of water in the tank was V m^3

In a model of this situation

- the sides of the tank have negligible thickness
- the rate of change of V is inversely proportional to the square root of h

(a) Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\lambda}{\sqrt{h}}$$

where λ is a constant.

(3)

Given that

- initially the height of the water in the tank was 1.44 m
- exactly 8 minutes after water started flowing into the tank the height of the water was 3.24 m

(b) use the model to find an equation linking h with t, giving your answer in the form

$$h^{\frac{3}{2}} = At + B$$

where A and B are constants to be found.

(5)

(c) Hence find the time taken, from when water started flowing into the tank, for the tank to be completely full.

(2)

a) $V = 20 \times 10 \times h$ $\qquad \frac{dV}{dt} = \frac{k}{\sqrt{h}}$

$V = 200h$

$\frac{dV}{dh} = 200$ (1)

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}} \quad \left(\text{where } \lambda = \frac{k}{200}\right)$$ (1) (1)

b) when $t=0$, $h=1.44$

when $t=8$, $h=3.24$

$$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \int h^{1/2}\,dh = \int \lambda\,dt$$ (1)

$$\frac{2}{3}h^{3/2} = \lambda t + c$$ (1)

sub in $t=0$, $h=1.44$

$$\frac{2}{3}(1.44)^{3/2} = 0\lambda + c \Rightarrow c = 1.152$$ (1)

sub in $t=8$, $h=3.24$

$$\frac{2}{3}(3.24)^{3/2} = 8\lambda + 1.152 \Rightarrow \lambda = 0.342$$ (1)

$$\frac{2}{3}h^{3/2} = 0.342t + 1.152$$

$$h^{3/2} = 0.513t + 1.728$$ (1)

c) tank is full when $h=5$

$$(5)^{3/2} = 0.513t + 1.728$$ (1)

$$t = \frac{5\sqrt{5} - 1.728}{0.513} \Rightarrow t = 18.4 \text{ minutes (3sf)}$$ (1)